Entanglement Criterion for Multi-Mode Gaussian States

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In this paper, we extend Simon's criterion for Gaussian states to the multi-mode Gaussian states using the Marchenko-Pastur theorem. We show that the Marchenko-Pastur theorem from random matrix theory as necessary and sufficient condition for separability.

The most important question in the quantum information is how does one say that the given state is entangled or separable? This question has been be answered by peres [1] in the finite dimensional case. Horodecki [2] has shown that this condition is necessary and sufficient for separability in the 2×2 and 2×3 dimensional cases, but fails for the higher dimensions. In the literature, this condition is known as Peres-Horodecki criterion. The Peres-Horodecki criterion for the covariance matrix using Wigner distribution has been given for the Gaussian states by Simon [3]. It has been shown in the same paper that the transpose operation T, which takes every $\hat{\rho}$ to its transpose $\hat{\rho}^T$, is *equivalent to* a mirror reflection in phase space:

$$\hat{\rho} \longrightarrow \hat{\rho}^T \iff W(q,p) \longrightarrow W(q,-p). \tag{1}$$

In this paper we extend the Simon's criterion to the ensemble of n-bipartite system of two modes using the Merchnko Pasture law in random matrix theory. The Marchenko-Pastur law appears in the famous Marchenko-Pastur theorem [4].

In random matrix theory, the dynamics of the ensemble of an infinite dimensional random matrix is described by the probability distribution function [5]

$$\mathcal{P}(\lambda_1, \ldots, \lambda_N)\, d\Lambda = c_n e^{-\beta H}\; d\Lambda \tag{2}$$

where the matrix H is brought into the following form $H = -\sum_{i=1}^{N} V(\lambda_i) - \sum_{i<1}^{N} ln|\lambda_i - \lambda_j|$ here $V(\lambda_i)$ is the potential and the λ_i are the eigenvalues with i being a free index running from $1, 2..N$, then $|\lambda_i - \lambda_j|$ is the Vandermonde determinant, $d\Lambda = d\lambda_1 \ldots d\lambda_N$, c_n is constant of proportionality and the index $\beta = 1, 2, 4$ characterises the real parameters of the symmetry class of orthogonal, unitary and symplectic respectively for the random matrix H.

The invariant ensembles in random matrix theory is classified into three class, the Gaussian ensembles these matrices are known as the Wigner matrices, the Wishart matrices and the two Wishart matrices. For details ref [6]. The Marcenko Pastur's Quarter-Circle Law is given for the Wishart matrices and they are defined as follows.

Namely, let $X_1, \cdots, X_n$ be independent and identically distributed random column vectors of $\mathcal{R}^m$, with mean 0 and covariance matrix I_m. Let us consider the $m \times m$ empirical covariance matrix

$$\Sigma = \frac{1}{n}\sum_{i=1}^{n} X_i X_i^T = \frac{1}{n} Y Y^T \text{where } Y = (X_1 \cdots X_n). \tag{3}$$

Then the distribution of the eigenvalues of Σ are considered under the different assumptions on the number of variables m and the number of observations n or m is the sample size and n is the dimension of the vectors. Their joint probability distribution function of eigenvalues is given by the Laguerre polynomials [6]. We have $E(\frac{1}{n}YY) = I_m$ and the strong law of large numbers indicates that with probability one,

$$lim_{n\to\infty} \frac{1}{n} Y Y^T = I_m. \tag{4}$$

In the case m grows with n, here m is the number of random variables and the n is the dimension of the Hilbert space, one has the following theorem known as the Marchenko-Pastur theorem [4].

Theorem: *Let us assume for simplicity that the components of X_i are Gaussian random variables. zero Mean, unit variance, and bounded moments that is there is some bound B, independent of m, such that $\forall n, E(|x_{ij}|^k) \leq B$. Then n depends on m in such a way that $m/n \to r \leq 1$ as $n \to \infty$. Under these assumptions, the distribution of the eigenvalues of $\frac{1}{n}YY^T$ asymptotically approaches the Marcenko-Pastur law as $n \to \infty$*

$$f(x) = \frac{\sqrt{(x-a)(b-x)}}{2\pi x r}. \tag{5}$$

where $a = (1-\sqrt{r})^2$ and $b = (1+\sqrt{r})^2$. For the case $r = 1$ it reduces to a famous quarter circle law.

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One important question that arises is how does one compare the covariance matrix constructed from the random matrices to a covarience matrix obtained from a definite state? First thing, if one looks at this two covarince matrices they are Gaussian. Then the question of randomness in the quantum system, this comes from the wave function. This is shown by one of the authors [7], the connection between the action equation for the stationary points of the random matrix ensemble $V(x) - \frac{1}{2}\sum_{k\neq l} log(|x_k - x_l|)$ where $V(x)$ is the potential and the quantum momentum function [8, 9] $p = \sum_{k=1}^{n} \frac{-i}{x-x_k} + Q(x)$ here the moving poles are simple poles with residue $-i\hbar$ (we take here $\hbar = m = 1$) and $Q(x)$ is the singular part of the quantum momentum function.

Therefore, if one takes this point of view that there is randomness associated with the wave function then one can see that the Wigner function or distribution is like the Wishart distribution. In random matrix theory the ensembles constructed using covariance matrix are the known as the Wishart Matrices and their joint probability distribution function of eigenvalues is given by the Laguerre polynomials [6]. The Wigner function for the Harmonic oscillator are the Laguerre polynomials [10]. It is shown in the reference [11] the Wigner function of the squeezed displaced vacuum state gives the Laguerre polynomials.

It is well known now that the squeezed states are non classical states and shows the property of entanglement [3]. The condition for squeezing is characterized in terms of covariance matrix using Wigner function reduces to a uncertainty relationship [12–14]

$$V + \frac{i}{2}\Omega \geq 0, \tag{6}$$

where V is the covariance matrix and is obtained by arranging the phase space variables and the hermitian canonical operators into four-dimensional column vectors

$$\xi = \left(\, q_1 \;\; p_1 \;\; q_2 \;\; p_2 \, \right), \quad \hat{\xi} = \left(\, \hat{q}_1 \;\; \hat{p}_1 \;\; \hat{q}_2 \;\; \hat{p}_2 \, \right).$$

Then the vector ξ satisfies the following commutation relationship [12]

$$\begin{aligned} [\hat{\xi}_\alpha, \hat{\xi}_\beta] &= i\,\Omega_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3, 4; \\ \Omega &= \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{7}$$

In particular, the condition for the multimode squeezing is also defined in terms of covariance matrix in ref [14].

Let us consider a ensemble n-bipartite system of two modes described by annihilation operators $\hat{a}_i = \sum_{i=1}^{n}(\hat{q}_{A_i} + i\,\hat{p}_{A_i})/\sqrt{2}$ and $\hat{b}_i = \sum_{i=1}^{n}(\hat{q}_{B_i} + i\,\hat{p}_{B_i})/\sqrt{2}$. By definition, a quantum state $\hat{\rho}$ of the bipartite system is separable if and only if $\hat{\rho}$ can be expressed in the form

$$\hat{\rho} = \sum_{A_i, B_i = 1}^{n} \sum_{j} p_j \, \hat{\rho}_{jA_i} \otimes \hat{\rho}_{jB_i}, \tag{8}$$

with *nonnegative* p_j's, where $\hat{\rho}_{jA_i}$'s and $\hat{\rho}_{jB_i}$'s are density operators of the modes of A and B respectively. To check that the given n-bipartite system of two modes is entangled or not we would perform that partial transpose operation on the density matrix (8) denoted PT. Then one calculates the eigenvalues partial transposed density matrix. If all the eigenvalues are positive then the it is a separable density matrix if it has atleast one eigenvalue then the state is entangled. This is the Peres-Horodecki separability criterion.

In the case of large matrices it is hard to check the positivity of the all eigenvalues, but the Marchenko-Pastur theorem in random matrix theory gives a condition for the convergence of the eigenvalues for the large matrices. This property of the random matrices is used to check weather a given state is separable or entangled.

In order to apply the Peres-Horodecki separability criterion for the Gaussian states one has to go over to phase space picture and then study the partial transpose operation in the Wigner picture, it is convenient to arrange the phase space variables as

$$\xi = \left(\, q_{A1} \;\cdots\; q_{An} \;\; q_{B1} \;\cdots\; q_{Bn} \;\; p_{A1} \;\cdots\; p_{An} \;\; p_{B1} \;\cdots\; p_{Bn} \, \right).$$

In terms of hermitian canonical operators

$$\hat{\xi} = \left(\, \hat{q}_{A1} \;\cdots\; \hat{q}_{An} \;\; \hat{q}_{B1} \;\cdots\; \hat{q}_{Bn} \;\; \hat{p}_{A1} \;\cdots\; \hat{p}_{An} \;\; \hat{p}_{B1} \;\cdots\; \hat{p}_{Bn} \, \right).$$

The commutation relations take the compact form[14]

$$\begin{aligned} [\hat{\xi}_\alpha, \hat{\xi}_\beta] &= i\,\Omega_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3, 4; \\ \Omega &= \begin{pmatrix} J & \cdots & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & \cdot & J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \end{aligned} \tag{9}$$

here Ω is a $4n \times 4n$ matrix. In the two mode case it has been shown in ref [3] under partial transpose operation on the bipartite density operator transcribes has the following transformation on the Wigner distribution:

$$PT: \quad W(q_1, p_1, q_2, p_2) \longrightarrow W(q_1, p_1, q_2, -p_2). \tag{10}$$

It is clear from the above that the sign momentum of the second is changing. Therefore when we apply partial transposition on multi-mode system is equivalent to flipping the sign of momentum variable of second mode that is p_{B_i}.

The Peres-Horodecki separability criterion for Gaussian state is defined in ref [3] as reads: *if $\hat{\rho}$ is separable, then its Wigner distribution necessarily goes over into a Wigner distribution under the phase space mirror reflection* Λ. $W(\Lambda\xi)$, like $W(\xi)$, should possess the "Wigner quality", for any separable bipartite state.

In multimode case, the covariance matrix will be of dimension $4n \times 4n$ that is dimension $2n$ from mode A and another dimension $2n$ from mode B. By following the procedure suggested in the ref [3] and arranging the uncertainties or variances into a $4n \times 4n$ real variance

matrix V, defined through $V_{\alpha\beta} = \langle\{\Delta\hat{\xi}_\alpha, \Delta\hat{\xi}_\beta\}\rangle$, where $\Delta\hat{\xi} = \hat{\xi} - \langle\hat{\xi}\rangle$, here $\langle\hat{\xi}_\alpha\rangle = \mathrm{tr}\hat{\xi}_\alpha\hat{\rho}$. Then the uncertainty principle [3, 13, 14].

$$V + \frac{i}{2}\Omega \geq 0. \tag{11}$$

Note that (6) implies, in particular, that $V > 0$, for details ref [3, 13, 14].

Now by using a real linear transformation on $\hat{\xi}$ specified by $4n \times 4n$ real matrix $S^{(r)}$

$$\hat{\xi} \to \hat{\xi}' = S^{(r)}\hat{\xi} \tag{12}$$

This transformation preserves the commutation relations [14]. We use $S^{(r)}$ in such a way that the

$$\hat{\xi}' = \left(\hat{q}_{A1} \;\; \hat{q}_{B1} \;\; \cdots \;\; \hat{q}_{An} \;\; \hat{q}_{Bn} \;\; \hat{p}_{A1} \;\; \hat{p}_{B1} \;\; \cdots \;\; \hat{p}_{An} \;\; \hat{p}_{Bn} \right).$$

In terms of annihilation and creation operators which is equivalent to

$$\hat{\xi}' = \left(\hat{a}_{A1} \;\; \hat{b}_{B1} \;\; \cdots \;\; \hat{a}_{An} \;\; \hat{b}_{Bn} \;\; \hat{a}^\dagger_{A1} \;\; \hat{b}^\dagger_{B1} \;\; \cdots \;\; \hat{a}^\dagger_{An} \;\; \hat{b}^\dagger_{Bn} \right).$$

Thus the covariance matrix is written as

$$V = \begin{pmatrix} \sigma_{11} & \cdots & \cdot & \sigma_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \sigma_{n1} & \cdots & \cdot & \sigma_{nn} \end{pmatrix}, \tag{13}$$

where σ_{ij} is a 2×2 matrix constructed from expectation values of $(a_{Ai}, b_{Bi}, a^\dagger_{Ai}, b^\dagger_{Bi},)$ elements. Since V is a bonifide the covariance matrix of Gaussian states then the eigenvalue distribution of this covariance matrix should follow Merchnko Pasture law. By identifying this covariance matrix Gaussian states with the Gaussian random covariance matrix we get that $m = 2n$ and $n = 4n$. One can clearly see that as $m \to \infty$ and $n \to \infty$ the ration $m/n = 1/2$. Therefore, the distribution of the eigenvalues of $V = \hat{\xi}'\hat{\xi}'^T$ asymptotically approaches the Marcenko-Pastur law as $m, n \to \infty$

$$f(x) = \frac{\sqrt{(x-a)(b-x)}}{\pi x}. \tag{14}$$

where $a = (1 - \sqrt{\frac{1}{2}})^2$ and $b = (1 + \sqrt{\frac{1}{2}})^2$. It is evident from eq (10) that under the partial transpose the Wigner distribution undergoes mirror reflection, and that reflects at the covariance matrix level by the following transformation $V \to \tilde{V} = \Lambda V \Lambda$. Then the new distribution function $W(\Lambda\xi)$ has to be a Wigner distribution if the state under consideration is separable, then one has

$$\tilde{V} + \frac{i}{2}\Omega \geq 0, \quad \tilde{V} = \Lambda V \Lambda, \tag{15}$$

where $\Lambda = (1, \cdots, 1, 1, -1, \cdots, 1, -1)$, there are total $4n$ ones, the first $2n$ the $+1$ corresponds to for the q_{Ai}, q_{Bi} in the second $2n$ the $+1$ corresponds to the p_{Ai} and the -1 corresponds to the p_{Bi}. In the covariance matrix this is equivalent to transposing each σ_{ij}. Hence one has

$$\tilde{V} = \begin{pmatrix} \sigma^T_{11} & \cdots & \cdot & \sigma^T_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \sigma^T_{n1} & \cdots & \cdot & \sigma^T_{nn} \end{pmatrix}, \tag{16}$$

Given a Gaussian states say $\hat{\rho}$ constructed form Hilbert space $H_A \otimes H_B$, under partial transposition of B, $\hat{\rho}$ has to be a bonifide density matrix. Then the Peres-Horodecki criterion is imposed on the covariance matrix constructed from the phase space variables [3]. Then the Peres-Horodecki criterion for the multimode continuous variables or the Simon's criterion for multimode reduces to the following theorem.

Theorem : *It is necessary and sufficient condition for separability for a given multimode Gaussian state if the eigenvalue distribution of the partially transposed covariance matrix $\tilde{V}$ satisfies Marcenko-Pastur law .*

Proof: Let us assume that the partial transposed covariance matrix $\tilde{V}$ is a bonifide covariance matrix of Gaussian states and the eigenvalue distribution of this covariance matrix satisfy's the Merchnko Pasture law. Thus by identifying this covariance matrix of Gaussian states with the Gaussian random covariance matrix we get that $m = 2n$ and $n = 4n$. One can clearly see that as $m \to \infty$ and $n \to \infty$ the ratio $m/n = 1/2$. Therefore, the distribution of the eigenvalues of $V = \hat{\xi}'\hat{\xi}'^T$ asymptotically approaches the Marcenko-Pastur law as $m, n \to \infty$

$$f(x) = \frac{\sqrt{(x-a)(b-x)}}{\pi x}. \tag{17}$$

where $a = (1 - \sqrt{\frac{1}{2}})^2$ and $b = (1 + \sqrt{\frac{1}{2}})^2$. This gives a condition that the eigenvalues x is bounded above and below by $2\sqrt{2} < x - 3 < -2\sqrt{2}$. Within this range the states are separable and outside they are entangled.

In Conclusion, we extend Simon's criterion for Gaussian states to the multi-mode Gaussian states using the Marchenko-Pastur theorem from random matrix theory as necessary and sufficient condition for separability.

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